Advances in Discrete Logarithm Computations

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Discrete logarithms

Discrete logarithms

- Given a multiplicative group G with generator g
- Computing discrete logarithms is inversing $n \to g^n$
- Hard in general and used as a hard problem in cryptography
- Algorithmic viewpoint
  - Generic algorithms (for any G)
  - Specific algorithms (make use of group representation)

Classical groups for Dlog in Cryptography

- Integers modulo p
- More general finite fields $\mathbb{F}_{p^k}$
- Elliptic curves over finite fields

Generic algorithms: Pohlig-Hellman

- Given a multiplicative group G with generator g
- Given $|G| = \prod_{i=1}^{k} p_i^{e_i}$
- To compute dlogs in G, it suffices to compute dlogs in:

$$G_i = \langle g^{|G|/p_i} \rangle \quad \text{(Group of order } p_i\text{)}$$

Generic algorithms: $|G| = p$

- There exist algorithms with complexity $O(\sqrt{p})$ to solve:

$$y = g^n$$

- Baby-step giant-step (let $R = \lceil \sqrt{p} \rceil$):
  - Create list $y, y/g, \cdots, y/g^{R-1}$
  - Create list $1, h, h^2, \cdots, h^{R-1}$, where $h = g^R$
  - Find collision
- Can be improved to memoryless algorithms using cycle finding techniques

Index calculus algorithms

- Relation generation phase
  - Choose small subset $S \subset G$ of "small elements"
  - Sparse multiplicative relation: Sequence $(s_1, e_1), \cdots (s_k, e_k)$ such that:

$$\prod_{i=1}^{k} s_i^{e_i} = 1$$

  - Each gives a sparse linear equation:

$$\sum_{i=1}^{k} e_i \log(s_i) = 0$$

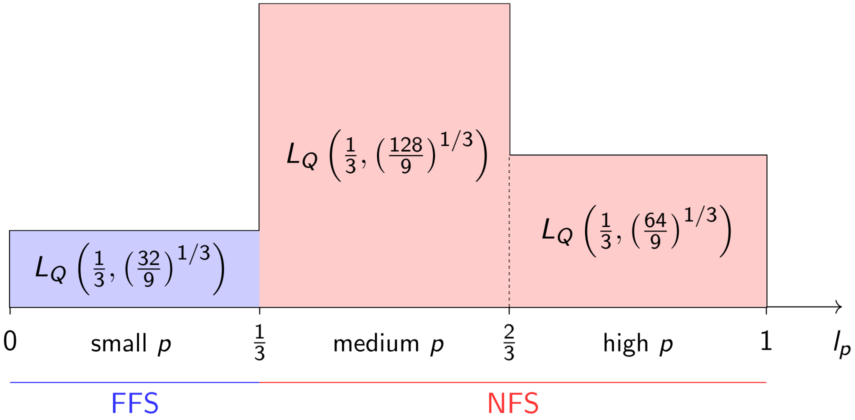
  - Modulo group order for discrete log

Index calculus algorithms

- Linear algebra phase
  - Large sparse system
  - Numbers of unknowns in range up to dozens of millions
  - Number of equations potentially very large
  - Need to use large computers to solve such systems
  - Often the limiting phase for practical computations
- Individual logarithm phase

Complexity of Index calculus algorithms (before 2013)

$$L_Q(\beta, c) = \exp((c + o(1))(\log Q)^{\beta}(\log\log Q)^{1-\beta}).$$



Discrete Logarithms, simplified FFS [JL06]

- To represent the finite field $\mathbb{F}_{p^k}$
- Choose two univariate polynomials f_1 and f_2
  - with degrees d_1 and d_2 and $d_1 d_2 \geq k$.
  - Such that $x - f_1(f_2(x))$ has:
    - an irreducible factor of degree k (modulo p).
- This defines the finite field by the relations:
  - $x = f_1(y)$ and $y = f_2(x)$

Commutative diagram

$$\begin{tikzcd} & \mathbb{F}_p[X,Y] \arrow[dl, "Y \mapsto f_2(X)"'] \arrow[dr, "X \mapsto f_1(Y)"] & \\ \mathbb{F}_p[X] \arrow[dr, "X \mapsto x"] & & \mathbb{F}_p[Y] \arrow[dl, "Y \mapsto y"] \\ & \mathbb{F}_{p^k} & \end{tikzcd}$$

Here, S contains low degree polynomials in X and Y.

Discrete Logarithms, simplified FFS [JL06]

- Optimal for $p = L_{p^k}(1/3)$

- Choose smoothness basis $S = \{x - \alpha, y - \alpha | \alpha \in \mathbb{F}_p\}$
- Consider elements:

$$\begin{tikzcd}[column sep=tiny] & xy + ay + bx + c \arrow[dl] \arrow[dr] & \\ x\,f_2(x) + af_2(x) + bx + c & & y\,f_1(y) + ay + bf_1(y) + c \end{tikzcd}$$

- When both sides split $\Rightarrow$ Relation
- Heuristic cost of finding relation (sieving):

$$(d_1 + 1)!\,(d_2 + 1)!$$

- Individual log. descent negligible compared to initial phase

Linear change of variables [J13]

- Further restrict to $y = x^{d_1}$
- Then:

$$xy + ay + bx + c \quad = \quad x^{d_1+1} + ax^{d_1} + bx + c$$

- Perform change of variable: $x = aX$, we get:

$$a^{d_1+1}(X^{d_1+1} + X^{d_1} + b \cdot a^{-d_1}(X + c/(ab)).$$

- Change of variable does not affect splitting property
- One good left-hand side $\Rightarrow$ $p-1$ good left-hand sides
- Amortized cost of relation reduced to

$$\left(\frac{(d_1+1)!}{p-1} + 1\right) \cdot (d_2+1)!$$

Linear change of variables [J13]

$$\begin{tikzcd}
& xy + ay + bx + c \arrow[dr] & \\
x^{d_1+1} + ax^{d_1} + bx + c \arrow[ur] & & y\,f_1(y) + ay + bf_1(y) + c \\
X^{d_1+1} + X^{d_1} + b \cdot a^{-d_1}(X + c/(ab)) \arrow[u] & &
\end{tikzcd}$$

Small characteristic

Small characteristic – Setting

- Use basefield $\mathbb{F}_q$
- Define extension field by a relation:

$$x^q = \frac{h_0(x)}{h_1(x)} \quad \text{or} \quad x = \frac{h_0(x^q)}{h_1(x^q)},$$

$$\begin{array}{c} \mathbb{F}_{q^k} = \mathbb{F}_q[\theta] \\ \big|\, k \\ \mathbb{F}_q \end{array}$$

  gives degree $k = \deg(I(x))$ extension, where $I(x)$ is a divisor of $h_1(x)x^q - h_0(x)$ or $h_1(x^q)x - h_0(x^q)$. Let θ be a root of I.
- We have a systematic relation:

$$\prod_{\alpha \in \mathbb{F}_q} (x - \alpha) \quad = \quad x^q - x. \tag{1}$$

Small characteristic – Basic idea (simplified)

- Substitute x by $\frac{A(\theta)}{B(\theta)}$ in (1) and multiply by $B(\theta)^{q+1}$:

$$B(\theta) \cdot \prod_{\alpha \in \mathbb{F}_q} (A(\theta) - \alpha\, B(\theta)) = B(\theta) \cdot A(\theta)^q - A(\theta) \cdot B(\theta)^q$$

- Moreover, after expanding the right-hand size, we find:

$$B(\theta) A\left(\frac{h_0(\theta)}{h_1(\theta)}\right) - A(\theta) B\left(\frac{h_0(\theta)}{h_1(\theta)}\right).$$

- Let D be the maximum degree of A and B and define:

$$[A, B]_D(X) = h_1(X)^D \left(B(X) A\left(\frac{h_0(X)}{h_1(X)}\right) - A(X) B\left(\frac{h_0(X)}{h_1(X)}\right)\right).$$

Small characteristic – Key equation

- Equation (1) after substitution can be rewritten as

$$\prod_{\alpha \in \mathbb{P}_1(\mathbb{F}_q)} (A(\theta) - \alpha\, B(\theta)) = \frac{[A,B]_D(\theta)}{h_1(\theta)^D}. \tag{2}$$

- The degree of $[A,B]_D$ is $\leq D \cdot (H+1)$, where $H = \max(\deg(h_0), \deg(h_1))$.
- Thus relations between degree D polynomials can be found easily.

Properties and simplification of $[A, B]_D(X)$

- $[A, B]_D$ is bilinear
- $[A, A]_D = 0$.
- In Equation (2), A and B can be assumed monic.
- Since $[A, B]_D = [A, B - A]_D$, we may also assume $\deg B < \deg A$.
- Assume $\deg A = D$ and $\deg B = D - 1$. Then, using bilinearity, one may reduce the coefficient of X^{D-1} in A to 0.
- In the sequel, we assume:

$$\begin{aligned} A(X) &= X^D + A_{D-2}(X) \text{ and} \\ B(X) &= X^{D-1} + B_{D-2}(X). \end{aligned}$$

Small characteristic – Choice of D

- If $D = 0$ then A and B are constants, thus $[A, B]_0 = 0$.
- If $D = 1$ then $A = X$ and $B = 1$ is the only choice.
- If $D = 2$ then $A = X^2 + \alpha$ and $B = X + \beta$: q^2 candidates
- If $D = 3$ then $A = X^3 + \alpha_1 X + \alpha_0$ and $B = X^2 + \beta_1 X + \beta_0$: q^3 candidates
- Cost of Linear algebra is at least $O(q^{2D+1})$.

$\Rightarrow$ Logs of Degree $\leq D$ Polynomials in θ

Individual Logarithms a.k.a. Descent

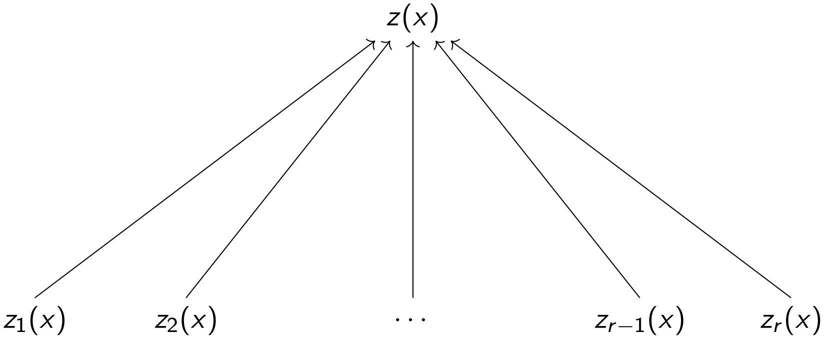
Descent strategies (Higher degree polynomial)

- Continued fractions (high degrees)
- Classical descent (for high to mid degrees, need subfield)
- Bilinear descent (for mid to low degrees)
- Quasi-polynomial descent (all degrees)
- ZigZag descent (all even degrees)

General principle

- Given target $z(x)$ in finite field, write:

$$z(x) = \prod_i z_i(x)^{e_i}, \quad \text{with smaller } z_i\text{s}$$



Continued fractions

- Given target $Z(x)$ find matrix:

$$\begin{pmatrix} A_1(x) & A_2(x) \\ B_1(x) & B_2(x) \end{pmatrix}, \text{such that}$$

$$Z(x) \equiv \frac{A_1(x)}{B_1(x)} \equiv \frac{A_2(x)}{B_2(x)} \pmod{I(x)}.$$

- With continued fraction or half-Gcd algorithms.
- Reduce degree by factor ≈ 2. Many representations:

$$Z(x) \equiv \frac{c_1(x)A_1(x) + c_2(x)A_2(x)}{c_1(x)B_1(x) + c_2(x)B_2(x)} \pmod{I(x)}.$$

Classical descent

- Need two variables x and y
- If $q = p^\ell$, let:

$$\begin{aligned} y &= x^{p^{\ell_1}} \qquad \text{then} \\ y^{p^{\ell_2}} &= x^{p^\ell} = \frac{h_0(x)}{h_1(x)}. \end{aligned}$$

- Let $F(x, y)$ be a (low degree) bivariate polynomial in $\mathbb{F}_q[x, y]$, then:

$$F(x, x^{p^{\ell_1}})^{p^{\ell_2}} = F(x^{p^{\ell_2}}, h_0(x)/h_1(x)) \quad \text{in } \mathbb{F}_{q^k}.$$

- Force $z(x)$ as divisor of $F(x, x^{p^{\ell_1}})$ or $F(x^{p^{\ell_2}}, h_0(x)/h_1(x))$ (linear algebra)
- Low arity in descent but can't go very low

New descents

- Remember Equation (2):

$$\prod_{\alpha \in \mathbb{P}_1(\mathbb{F}_q)} (A(\theta) - \alpha\, B(\theta)) = \frac{[A, B]_D(\theta)}{h_1(\theta)^D}.$$

- Make $z(x)$ appear on the right or left

Bilinear descent

- Search for k_1 and k_2 such that:

$$z(x) | [k_1, k_2]_D(x)$$

- Then $z(x)$ appears on the right in Equation (2).
- Arity $\approx q$ in descent

How to find k_1 and k_2 ?

- Algebraic approach : divisibility condition as a bilinear system
  - In general, use Groebner bases
  - For low-degree, it degenerates into easy linear algebra
- **Open problem:**
  Is there a more direct/efficient general approach ?
  Partial answer: Degree $2D$ to degree D a.k.a ZigZag [GKZ14]

Quasi-polynomial descent

- Make $z(x)$ appear on the right in the term:

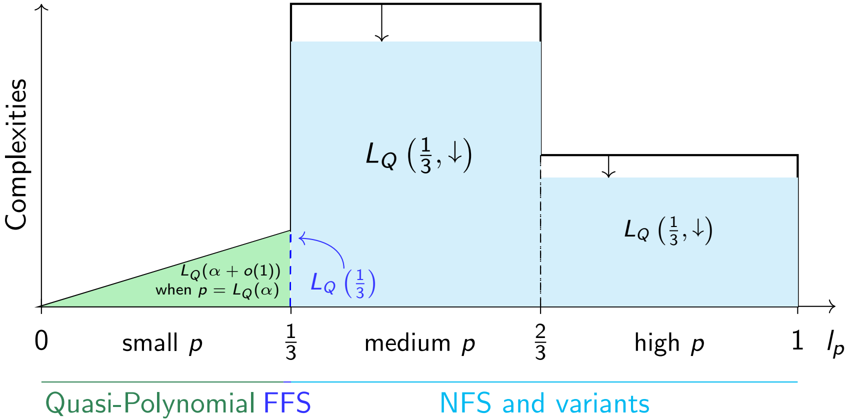
$$\prod_{\alpha \in \mathbb{P}_1(\mathbb{F}_q)} (A(\theta) - \alpha\, B(\theta))$$

- Choose $A(x) = z(x) + \alpha$ and $B(x) = x + \beta$
- Gives $\approx q^2$ equations.
- Simultaneous descent of all $z(x) + \lambda_1\, x + \lambda_0$
- Requires extra linear algebra step
- Arity q^2 in descent

Descent Tree

- Continued fractions, **at most one application**
- Classical descent, **many levels possible**
- Bilinear descent (or [GKZ14]), **in practice 4-5 levels max.**
- Quasi-polynomial descent **in practice 2 levels max.**

Complexities of Index Calculus Algorithms



Conclusion

Questions ?

How to find k_1 and k_2 ?

- Algebraic approach : divisibility condition as a bilinear system
  - In general, use Groebner bases
  - For low-degree, it degenerates into easy linear algebra
- Lattice reduction approach :
  - Further assume that k_1 and k_2 split into linear term
  - Since $z(x)$ is irreducible, it encodes a finite field
  - Take logarithms of elements :

$$\frac{x-\alpha}{h_0(x)/h_1(x)-\alpha}, \quad \text{for } \alpha \in \mathbb{F}_q.$$

  - Find low weight sum of logarithms equal to 0
- **Open problem:**
  Is there a more direct/efficient general approach ?
  Partial answer: Degree $2D$ to degree D [GKZ14]